WARSAW SCHOOL OF ECONOMICS

Collegium of Economic Analysis

Summary of doctoral dissertation:

The Solow and the Mankiw-Romer-Weil models with endogenous savings rates. Analysis of chaotic dynamics

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I Justification for the choice of subject

"ONCE you start thinking about growth, it's hard to think about anything else."

Robert Lucas[1]

The theory of economic growth, present in the economy since 1930s, is based on mathematical models defined by a set of parameters and variables, which describe *"sources, mechanisms and processes of economic growth in a deterministic way"*[2].

A unique place among numerous models of growth is occupied by non-linear models, especially those, which explain the causes of economic fluctuations (cycles). In spite of wide range of possibilities, as a starting point for considering economic growth, the Solow (Solow-Swan[3]) model is still widely used. It's most known modification is a two-dimensional model developed in 1992 by N. Gregory Mankiw, David Romer and David N. Weil[4], which takes into account the additional factor of production that represents human capital. The Mankiw, Romer and Weil model is modified and used to explain the phenomenon of convergence both in stochastic[5] and deterministic versions. The introduction of endogenous factors into the model extends the scope of its application in theoretical considerations. Modifications take in account for example technological knowledge as yet another factor of production[6], technology transfer between countries[7], interdependencies between countries[8] or changes of production functions[9].

[1] Lucas R.E., On the Mechanics of Economic Development, „Journal of Monetary Economics", 1988, no 22, s. 3–42.
[2] Malaga K., O niektórych dylematach teorii wzrostu gospodarczego i ekonomii, „ZK Polskie Towarzystwo Ekonomiczne", Warszawa 2009.
[3] In February 1956, R.M. Solow in *The contribution to the theory of economic growth* published in the "Quarterly Economic Journal" presented the assumptions of his economic growth model. In November of the same year T.W. Swan described a very similar model in the article: *Economic growth and capital accumulation* published in the "Economic Register".
[4] Mankiw N. Gregory, David Romer, David N. Weil, *A Contribution to the Empirics of Economic Growth*, „Quarterly Journal of Economics", 107, 1992, s. 407 – 437.
[5] Kevin Lee K., Pesaran M.H., Smith R., *Growth and Convergence in a Multi-Country Empirical Stochastic Solow*, „Journal of Applied Econometrics", Vol. 12, No. 4 (Jul. - Aug., 1997), pp. 357-392.
[6] Nonneman W., Vanhoudt P., *A Further Augmentation of the Solow Model and the Empirics of Economic Growth for OECD Countries*, „The Quarterly Journal of Economics", Vol. 111, No. 3 (Aug., 1996), pp. 943-953.
[7] Dowrick S., Rogers M. *Classical and Technological Convergence: Beyond the Solow-Swan Growth Model*, „Oxford Economic Papers", Vol. 54, No. 3 (Jul., 2002), pp. 369-385.

Multidimensional versions of this model were also discussed[10].

The subject raised in this study is a continuation and deepening of problems discussed in the research aforementioned. The motivation for taking into account variable interest rates of savings in analyzed models, were results of studies carried out in 2001 by BS Bernanke and R.S Gurkaynak, according to which a long-term growth should be correlated with the behavioural variables, which impact on savings rates[11].

II Objective of dissertation, thesis and research hypotheses

The main objective of this study is to analyse and compare dynamics of the same non-linear economic growth model in two variants, which differ in modelling of independent variable, namely time. Taking the proposed models as an example, the influence of time variable character on system behaviour has been presented. Furthermore, conditions favouring the occurrence of economic cycles were indicated, as well as circumstances due to which the use of endogenous savings rates impacts periodic capital fluctuations. The model was extended with human capital, influence of this extension on global system dynamics was presented. - Solow model's and Mankiw-Romer-Weil model's modification with variable savings rates according to Richard H. Day[12] - is the original contribution of the author. Numerical analysis of proposed models was performed using iDMC software[13], distributed under terms of the GNU General Public License.

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[8] Ertur C., Koch W., *Growth, Technological Interdependence and Spatial Externalities: Theory and Evidence*, „Journal of Applied Econometrics", Vol. 22, No. 6 (Sep. - Oct., 2007), pp. 1033-1062.

[9] Masanjala W.H., Papageorgiou C. *The Solow Model with CES Technology: Nonlinearities and Parameter Heterogeneity*, „Journal of Applied Econometrics", Vol. 19, No. 2 (Mar. - Apr., 2004), pp. 171-201.

[10] Tokarski T., *Optymalne stopy inwestycji w N-kapitałowym modelu wzrostu gospodarczego*, „Gospodarka Narodowa", Nr 9/2007.

[11] Bernanke B.S., Gurkaynak R.S. *Is Growth Exogenous? Taking Mankiw, Romer and Weil Seriously*, NBER Working Paper No. 8365, July 2001.

[12] Day R.H., *Irregular Growth Cycles*, „The American Economic Review", Vol. 72, No. 3 (Jun.,1982), 406-414.

[13] iDMC - interactive dynamical model calculator, M.Lines, A.Medio, www.gitorious.org/idmc

The research thesis put forward in this study is the possibility of improving the quality of forecasts provided by deterministic models of economic growth, by taking into account characteristic behaviour for non-linear dynamic systems.

As part of the thesis put forward in this way, following research hypotheses were verified:

• Choosing a time variable as a discrete or continuous variable can affect the dynamics of the non-linear model used significantly.
• The introduction of yet another dimension into the non-linear system (e.g. representing human capital) changes dynamic properties of the model (including the scope of forecasting).
• The use of endogenous savings rates in the non-linear growth model has a significant impact on the dynamics of model and can induce cycles in the economy.

III Review of literature

Among factors affecting the economy, we find both demand for consumption and investment goods (articles and services), as well as supply factors (e.g work, capital, land, raw materials)[14]. Depending on the time perspective, we can see a significant impact of the components of global demand (in short run) or supply factors of growth (in long run). The Mankiw-Romer-Weil model used in the dissertation includes an additional (in reference to the original Solow model) supply factor - human capital.

The interest in human capital and its impact on the level of wealth (prosperity) has lasted for several centuries. William Petty (1623-1687) as the first one took up considerations of this type[15], and he noticed that the capital inherent in people is characterized by many similarities with fixed capital (tangible)[16]. The authors of the concept and theory of human capital are T.W Schultz[17], G.S Becker[18]

[14] Begg D., Fischer S., Dornbusch R., *Makroekonomia*, Polskie Wydawnictwo Ekonomiczne, Warszawa 2003
[15] Kunasz M., *Teoria kapitału ludzkiego na tle dorobku myśli ekonomicznej.* (w:) A. Manikowski, A. Psyk (red.). „Unifikacja gospodarek europejskich: szanse i zagrożenia". Uniwersytet Warszawski, Warszawa 2004, s.28.
[16] Domański S.R., *Kapitał ludzki i wzrost gospodarczy*, PWN, Warszawa 1993.
[17] Schultz T.W., *Investment in human capital*, „American Economic Review" 1961, 51 (1), s. 1– 7.
[18] Becker G.S., *Human capital: A theoretical and empirical analysis*, New York: Columbia University Press 1964.

and J. Mincer[19]. In the 1980s of the XX century, the concept of "human capital" appeared in the theory of endogenous growth, then in 1992 in the Mankiw-Romer-Weil model. Since then, a number of studies dedicated to the direct or indirect presence of human capital in the theory of growth have been published, indicating it's meaning for theoretical and empirical research[20].

The obvious consequence of the objective of modelling (simplifying economic reality) seems to be the creation of linear deterministic models. However, the linearity of system implies the existence of equilibrium state or discrepancy, towards which the system evolve. At this point, the first imperfection of linear models is revealed. The interest of economists is focused only on the interpretation[21] of the equilibrium state conditions, because other borderline behaviours of the model are not useful from the economic perspective - they can mean for example the crash of economy or negative prices.

However, economic systems are subject to the activity of such a large number of factors that it is justified to use stochastic models, especially that such models are often well matched to actual data. It is worth considering: "does the assessment of matching itself lead to abuse of models of a particular class?" This question is posed by Witold Orzeszko (2005)[22] and based on Lorenz's considerations[23], using non-linear Hicks model with limitations he demonstrates, that "the concept of adding noise to a linear model can lead to a misidentification of the mechanism of economic phenomenon studied."

The analysis of literature on economic growth theory shows that a reasonable alternative to stochastic linear models are non-linear deterministic systems, present in economy since mid-twentieth century, in particular used to describe economic

[19] Mincer J., Investment in Human Cápital and Personal Income Distribution, "The Journal of Political Economy" 1958, vol. 66, no. 4, ss. 281-302.
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[21] Bullard J.B., Butler A. , *Nonlinearity and Chaos in Economic Models: Implications for Policy Decisions*, maszynopis 1991-002B, The Federal Reserve Bank of St. Louis., s. 5.
[22] Orzeszko W., *Modele chaotyczne w ekonomii*, „Acta Universitatis Nicolai Copernici. Nauki Humanistyczno-Społeczne. Ekonomia", Tom 36 (2005) s. 155-170, Toruń 2005.
[23] Lorenz H.W., *Nonlinear Dynamical Economics and Chaotic Motion*, Springer – Verlag, Berlin 1989, Heidelberg, s. 29.

cycles (Kalecki, Timberg, Kaldor)[24]. However, even they (mainly early non-linear models with a regular course) do not describe complex nature of economic reality characterized by variety of potential behaviours. This encourages economists to pay attention to another class of non-linear models, which due to their complex evolution, are similar to random systems, in particular they generate cycles with variable period and amplitude.

In the same year (1975), in which the term "deterministic chaos" appeared in literature (Li and York)[25], May and Beddington[26] signalled the possibility of applying chaos theory to economy. Since then, many new economic models have been constructed with chaotic dynamics or chaos has been recognized in existing models[27]. Models of this type are now successfully used to explain economic cycles, and in modern systems the root cause of cycles are most often time delays in agents' reactions to market signals and their changing expectations regarding future[28]. The starting point for these considerations are usually linear models (in the original version) then modified to a form which is characterized by a more complex dynamics. For example, the Samuelson-Hicks model is very popular (interaction of multiplier and accelerator), the non-linear versions of which are considered in the studies: Gabisch (1984)[29], Hommes, Nusse (1990)[30], Hommes (1991)[31]; Kruszewski (2006[32], 2010[33]), Puu, Gardini, Sushko (2005)[34]; PUU (2007)[35]. The analysis of studies,

[24]Drabik E.: *Dynamiczne nieliniowe modele ekonometryczne: model cykli koniunkturalnych Kaleckiego-Kaldora oraz model wzrostu*. W: *Rynek kapitałowy. Skuteczne inwestowanie*. Red. W. Tarczyński. Szczecin 2002, s. 261-273.
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[26] May R., Beddington J.R., *Nonlinear Difference Equations: Stable Points, Stable Cycles, Chaos*. Maszynopis, 1975.
[27] Miśkiewicz-Nawrocka M., *Modele ekonomiczne z dynamiką chaotyczną* Wydawnictwo Uniwersytetu Ekonomicznego w Katowicach Czasopismo Studia Ekonomiczne, Rocznik 2013, Tom 132, Strony 56-66 oraz Orzeszko W. *Modele…, op.cit.*
[28] Grandmont J.M., *On endogenous competitive business cycles*, Econometrica 1985, vol. 50, 1345-1370.
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[31] Hommes C. H., *Adaptive learning and roads to chaos. The case of the cobweb model*, „Economic Letters", 36 (1991).
[32] Kruszewski R., *Periodic and quasi-periodic dynamics in a modified Samuelson-Hicks model*, Macromodels 2006, Łódź, 2007.
[33] Kruszewski R., *Wybrane modele współdziałania mnożnika i akceleratora. Analiza chaotycznej dynamiki*, Oficyna Wydawnicza SGH, Warszawa 2010.

in which non-linear (including chaotic) systems were used (among others: Cars Hommes[36], Marji Lines and Frank Westerhoff[37], Hans-Walter Lorenz[38], Tomasz Tokarski[39], Robert Kruszewski[40]) leads to conclusion that use of chaotic dynamics in the economy is purposeful and confirms that such models will continue to be successfully used to describe economic reality. Undoubtedly they will not replace (currently commonly used) stochastic linear models, but they are an important complement to them.

IV. Variable savings rates

Both Solow model and Mankiw-Romer-Weil model are characterized by exogenous, fixed savings rates. In models proposed by author, this restrictions have been removed. The savings rates are variable in time and endogenous. This postulate was implemented through the adaptation of R.H. Day[41] proposal of binding savings rate with income, assets owned and real interest rate.

Thus the savings are described by the equation:

$$sy_t = a\left(1-\frac{b}{r}\right)k_t, \qquad (1)$$

[34] Puu T., Gardini L., Sushko I., *A multiplier - accelerator model with floor determined by capital stock*, *Journal of Economic Behavior and Organization*, 56 (2005).
[35] Puu T., *The Hicksian trade cycle with floor and ceiling dependent on capital stock*, Journal of Economic Dynamics and Control, 31 (2007).
[36] Hommes C. H., *Adaptive learning and roads to chaos. The case of the cobweb model*, „Economic Letters", 36 (1991); Hommes C.H., *Periodic, almost periodic and chaotic behaviour in Hicks' non-linear trade cycle model*, „Economics Letters", 41 (1993) oraz Hommes C.H., *A reconsideration of Hicks' non-linear trade cycle model*, „Structural Change and Economic Dynamics", 6 (1995).
[37] Lines M., *Bifurcation scenarios in a heterogeneous agent, multiplier accelerator model*, „Pure Mathematics and Applications", 16 (2007); Lines M., Westerhoff F., *Infation expectations and macroeconomic dynamics: The case of rational versus extrapolative expectations*, „Journal of Economic Dynamics and Control", 34 (2010); Westerhoff H., *Nonlinear expectation formation, endogenous business cycles and stylized facto*, „Studies in Nonlinear Dynamics and Econometrics", 10 (2006c), Issue 4, Article 4.
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[41] Day R.H., *op.cit.*

Where r is the real interest rate: $r = \frac{df(k)}{dk}$,

whereas $a > 0$ i $b \in (0,r)$ are parameters that are behavioral in nature and depict consumer behavior in relation to the real interest rate (averaged values for the entire population).

Regarding the above assumption in the Mankiw-Romer-Weil model, we get:

$$s_K y_t = a_K\left(1 - \frac{b_K}{r_K}\right)k_t, \tag{2}$$

$$s_H y_t = a_H\left(1 - \frac{b_H}{r_H}\right)h_t, \tag{3}$$

$r_K = \frac{\partial \varphi(k,h)}{\partial k}$ and $r_H = \frac{\partial \varphi(k,h)}{\partial h}$

denotes interest rates equal to the marginal productivity of each type of capital, whereas $a_K, a_H > 0$, $b_K \in (0, r_K)$, $b_H \in (0, r_H)$ are behavioral parameters illustrating consumer behavior in relation to real interest rates (averaged values for the whole population).

Without loss of generality it was assumed in the study that the rates of the depreciation of physical and human capital in the Mankiw-Romer-Weil model are equal. This has simplified the algebraic side of transformations.

Likewise: $a_K = a_H$ and $b_K = b_H$.

The considerations used production function of Cobb-Douglas, where *h* and *k* are, respectively: physical and human capital per unit of effective labour.

Then: $f(k) = k^\gamma$, for the Solow model and $\varphi(k,h) = k^\alpha h^\beta$ for the Mankiw-Romer-Weil model, where $0 < \gamma < 1$, $0 < \alpha < 1$, $0 < \beta < 1$, $\alpha + \beta < 1$.

After the modification, i.e. the introduction of variable savings rates, according to: (1), (2), (3), the savings in both analyzed models are income-dependent and grow with the increase of the real interest rate.

V Dynamics of the Solow model and the Mankiw-Romer-Weil model with endogenous savings rates in discrete time

After applying the variable savings rate, according to (1), (2), (3), the equation describing the dynamics of capital per unit of effective labour in the Solow model takes the form:

$$k_{t+1} = \frac{1}{(n+1)(g+1)}\left[a(1-b\gamma^{-1}k_t^{1-\gamma})k_t + (1-\delta)k_t\right].$$

and the equations which describe the evolution in time of physical and human capital per unit of effective labour in the Mankiw-Romer-Weil model:

$$k_{t+1} = \frac{1}{(n+1)(g+1)}\left[a(1-b\alpha^{-1}k_t^{1-\alpha}h_t^{-\beta})k_t + (1-\delta)k_t\right],$$

$$h_{t+1} = \frac{1}{(n+1)(g+1)}\left[a(1-b\beta^{-1}k_t^{-\alpha}h_t^{1-\beta})h_t + (1-\delta)h_t\right].$$

The proposed models assumed a fixed, exogenous rate of population growth $n > 0$ and a constant exogenous rate $g > 0$ of labor-augmenting technology. Capital depreciates at the rate $\delta \in (0,1)$.

The dynamics of models constructed in this way was examined, in particular by determining the stationary state and conditions at which the equilibrium state was locally asymptotically stable. A detailed description of system's behaviour according to bifurcation theory, the Lapunov exponent and (in two-dimensional case) entropy of Kolmogorov and dimension of the Lapunov chaotic attractor was presented.

VI Dynamics of the Solow model and the Mankiw-Romer-Weil model in a continuous time with endogenous savings rates

After applying the variable savings rate, according to (1) the equation describing the dynamics of capital per unit of effective labour in the Solow model takes the form:

$$\dot{k} = a(1-b\gamma^{-1}k^{1-\gamma})k - (n+g+\delta)k$$

Again the proposed model assumed a fixed, exogenous rate of population growth $n>0$ and a constant exogenous rate $g>0$ of labor-augmenting technology. Capital depreciates at the rate $\delta\in(0,1)$.

Thus, like the original Solow model, the rate of changes in the capital stock per unit of effective work is the difference between the two expressions.

The term $sf(k)=a(1-b\gamma^{-1}k^{1-\gamma})k$ denotes actual (gross) investment, the term $(n+g+\delta)k$ denotes investment that goes to replacement of reduction in capital caused by depreciation, population growth and technological progress.

After applying the variable savings rate, according to (2) and (3) to the Mankiw-Romer-Weil model, we have got equations for the model that governs its dynamic:

$$\dot{k}=a(1-b\alpha^{-1}k^{1-\alpha}h^{-\beta})k-(n+g+\delta)k,$$

$$\dot{h}=a(1-b\beta^{-1}k^{-\alpha}h^{1-\beta})h-(n+g+\delta)h.$$

The analysis of the dynamics of models constructed in this way was performed. The equilibrium state and the conditions for its stability were determined. Both modified models (one- and two-dimensional) have exactly one locally asymptotically stable equilibrium state. In case of the low level of savings identified by the relationship proposed by R.H. of Day, the actual savings are fewer than the replacement ones, thus the capital is being comsumed.

VII Summary

Modified Solow and Mankiw-Romer-Weil models were presented in the dissertation. The modification involved introduction of variable savings rate of endogenous character, as proposed by R.H. Day[42]. Next, the dynamics of models constructed in this way in a continuous and discrete versions were examined. The most important conclusions resulting from the analyses (in relation to the above hypotheses) in the dissertation can be summarized as follows:

• Proposed models with a continuous time (both in one- and two-dimensional version) have exactly one stationary state and this is one locally asymptotically stable equilibrium state. Adoption variable savings rates in the way indicated by R. Day

[42] Day R.H., *op.cit.*

influenced not only the value of both types of capital in equilibrium state, but changed the character of capital dependence in the steady state on consumer's propensity to save. In original models (Solow and Mankiw-Romer-Weil) the dependence of capital (physical in the Solow model, physical and human in the Mankiw-Romer-Weil model) was a convex function of the savings rate in the stationary state. In modified models, the dependence of capital in stationary state on the parameter expressing the propensity of consumers to save is a concave function. This indicates the appearance of the "saturation" state for large values of the above parameter. To sum up: the introduction of variable savings rates has significantly changed the dependence of accumulated capital on consumers' propensity to save.

• Models modified by the introduction of a variable savings rate analysed in discrete time, despite the low number of dimensions are characterized by complex dynamics, and the degree of its complexity depends on the parameter illustrating consumer`s behaviour in relation to actual interest rates. In both cases (one- and two-dimensional model), the system lost its stability because of the period doubling bifurcation. Periodic, quasi-periodic, and chaotic behaviours appeared. The appearance of the deterministic chaos phenomenon implied the limitation of the forecasting time horizon to the so-called characteristic Lapunov time. Between the chaotic areas, windows of stability with low period appeared (high in Sharkovsky's order), interrupted by outbursts of intermittence, which occurred more and more frequently as the parameter representing consumers' propensity to save increased. In case of the model with human capital, the Neimark-Sacker bifurcation appeared with characteristic, quasi-periodic solutions.

• The extension of the Solow model with human capital, significantly limits the area of the stability of stationary state and the possibility of cycles of any periods. It has been demonstrated that in the proposed models the analysis of dynamics is possible and purposeful also outside the area of local asymptotic stability. Encouraging consumers to specific behaviour in relation to actual interest rates makes it possible to avoid chaos and even control occurrence and period of economic cycle.

• The decision on selecting a time variable (continuous or discrete) in case of non-linear models affects the applicability of model to specific purposes. For example, in case under consideration, the consequence of choosing a discrete time variable turned out to limit time of forecasting for specific model parameters, but the advantage of such a choice was the possibility to predict the appearance of economic cycles and their period.

The analysis carried out does not exhaust the subject, but shows that the modification of the deterministic economic growth model by the assumption of the variability of non-linear savings rates enriches the model's dynamics with behaviours characteristic of non-linear dynamical systems, increasing the quality of forecasts obtained using the model modified in such way.

Due to the complex form of some dependencies, and also due to the purpose of study and the need to focus on basic properties of models, a number of simplifications were adopted, e.g. it was assumed that the selected parameters were equal for both types of capital. Eventual continuation of study should start with the repeal of these assumptions.

The proposed models exhibit variable savings rates as a potential factor in the appearance of periodic fluctuations in capital, and thus can be a bridge connecting the theory of growth with the theory of the business cycle.

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